

# On the $P_1$ property of sequences of positive integers

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**Abstract.** It is well-known that for any non-constant polynomial  $P$  with integer coefficients the sequence  $(P(n))_{n \in \mathbb{N}}$  has the property that there are infinitely many prime numbers dividing at least one term of this sequence. Certainly, there is a proof based on the Chinese Remainder Theorem. In this paper we give proofs of two analytic criteria revealing this property of sequences.

## 0.1 Introduction

**DEFINITION 0.1.** We say that a sequence  $(n_k)_{k \in \mathbb{N}}$  of positive integers has the  $P_1$  property (we write  $(n_k)_{k \in \mathbb{N}} \in P_1$ ) if there exist infinitely many prime numbers dividing at least one term of this sequence. The main results of this paper are the following two theorems.

**THEOREM 0.1.** If  $(n_k)_{k \in \mathbb{N}}$  is an increasing sequence of positive integers and

$$\liminf_{k \rightarrow \infty} \frac{\ln(\ln(n_k))}{\ln(k)} = 0$$

then  $(n_k)_{k \in \mathbb{N}} \in P_1$ .

**THEOREM 0.2.** If  $(n_k)_{k \in \mathbb{N}}$  and  $(m_k)_{k \in \mathbb{N}}$  are increasing sequences of positive integers such that  $\gcd(n_k, n_{k+l}) < m_l$  for all positive integers  $k$  and  $l$  then  $(n_k)_{k \in \mathbb{N}} \in P_1$ .

## 0.2 Proof of theorem 1

Suppose we are given positive numbers  $w_1, w_2, \dots, w_n$  where  $n \in \mathbb{N}$ .

**DEFINITION 0.2.** For any  $W > 0$  define  $N(W; w_1, w_2, \dots, w_n) = \text{card}\{(k_1, k_2, \dots, k_n) | k_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^n k_i w_i \leq W\}$ .

We will use the inequality

$$N(W; w_1, w_2, \dots, w_n) \leq \frac{(W + \sum_{i=1}^n w_i)^n}{n! \prod_{i=1}^n w_i}$$

mentioned in [5].

*Proof.* Now suppose  $(n_k)_{k \in \mathbb{N}} \notin P_1$ . So there is a finite set  $S = \{p_1, p_2, \dots, p_n\}$  consisting of prime numbers such that each term of  $(n_k)_{k \in \mathbb{N}}$  is a product of some, not necessary distinct elements from  $S$ .

**DEFINITION 0.3.** For each  $l \in \mathbb{N}$  let us define  $t_l = \text{card}\{k | 1 \leq k \leq n, n_k \leq l\}$ .

Hence  $t_l \leq \text{card}\{(k_1, k_2, \dots, k_n) | k_i \geq 0, 1 \leq i \leq n, \prod_{i=1}^n p_i^{k_i} \leq l\}$  which is equivalent to  $t_l \leq \text{card}\{(k_1, k_2, \dots, k_n) | k_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^n \ln(p_i) k_i \leq \ln(l)\}$  where the latter number is  $N(\ln(l), \ln(p_1), \ln(p_2), \dots, \ln(p_n))$  due to definition 2.

Since

$$N(W; w_1, w_2, \dots, w_n) \leq \frac{(W + \sum_{i=1}^n w_i)^n}{n! \prod_{i=1}^n w_i}$$

we therefore have that there is  $c > 0$  such that

$$N(W; w_1, w_2, \dots, w_n) \leq cW^n$$

for all  $W > \delta > 0$ . Consequently for some  $a > 0$

$$t_l \leq N(\ln(l), \ln(p_1), \ln(p_2), \dots, \ln(p_n)) \leq a(\ln(l))^n$$

for all  $l \in \mathbb{N}$ ,  $l \geq 2$ .

If we substitute  $l = n_k$  for  $k = 2, 3, \dots$  we will get that

$$k = t_{n_k} \leq a(\ln(n_k))^n$$

hence

$$\ln(k) \leq \ln(a) + n \ln(\ln(n_k)), \quad k = 2, 3, \dots$$

Thereby

$$\liminf_{k \rightarrow \infty} \frac{\ln(\ln(n_k))}{\ln(k)} \geq 1/n > 0$$

which is a contradiction.

So,  $(n_k)_{k \in \mathbb{N}} \in P_1$  and the theorem is proved.  $\square$

**COROLLARY 0.1.** For any non-constant polynomial  $P$  with integer coefficients the sequence  $(P(n))_{n \in \mathbb{N}} \in P_1$ .

*Proof.* The sequence  $(P(n))_{n \in \mathbb{N}}$  is eventually monotone and

$$\lim_{k \rightarrow \infty} \frac{\ln(\ln(P(k)))}{\ln(k)} = 0.$$

It remains to use theorem 1.  $\square$

### 0.3 Proof of theorem 2

*Proof.* Suppose  $(n_k)_{k \in \mathbb{N}} \notin P_1$ . So there is a finite set  $S = \{p_1, p_2, \dots, p_s\}$  consisting of prime numbers  $p_1 < p_2 < \dots < p_s$  such that each term of  $(n_k)_{k \in \mathbb{N}}$  is a product of some, not necessary distinct elements from  $S$ . Since  $(n_k)_{k \in \mathbb{N}}$  is increasing it is unbounded hence there is at least one  $p \in S$  such that  $(\nu_p(n_k))_{k \in \mathbb{N}}$  is unbounded, where  $\nu_p(m) = \max\{k : p^k | m\}$  for any integer  $m$  and prime number  $p$ . WLOG we may assume that the set of such primes  $p$  is  $\{p_1, p_2, \dots, p_l\}$ , for some  $1 \leq l \leq s$ .

DEFINITION 0.4. For each  $1 \leq t \leq l$  and  $M \in \mathbb{N}$  we define

$$A_t(M) = \{k | \nu_{p_t}(n_k) > M\} = (s_{t,j})_{j \in \mathbb{N}}$$

.

COROLLARY 0.2.

$$\mathbb{N} = \bigcup_{t=1}^l A_t(M) \cup A_M,$$

for some finite set  $A_M$ . Moreover,  $A_M = \{k | \nu_{p_t}(n_k) \leq M, t = 1, 2, \dots, l\}$ .

Let us choose  $M$  large enough to satisfy  $2^M > m_l$ .

LEMMA 0.1.  $s_{t,j+1} - s_{t,j} > l$  for all  $t \in \{1, 2, \dots, l\}$  and  $j \in \mathbb{N}$ .

*Proof.* One has that  $p_t^M | n_{s_{t,j+1}}$  and  $p_t^M | n_{s_{t,j}}$ , so

$$m_{(s_{t,j+1}-s_{t,j})} > \gcd(n_{s_{t,j+1}}, n_{s_{t,j}}) \geq p_t^M \geq 2^M > m_l$$

hence  $s_{t,j+1} - s_{t,j} > l$  as desired.  $\square$

Therefore, for any  $t \in \{1, 2, \dots, l\}$  and  $N \in \mathbb{N}$  there are at most  $(\lfloor \frac{N}{l+1} \rfloor + 1)$  elements of  $A_t(M)$  in  $\{1, 2, \dots, N\}$ . Hence for each  $N$  there are at least

$$\Delta(N) = N - l(\lfloor \frac{N}{l+1} \rfloor + 1) > \frac{N}{l+1} - l$$

elements of  $\{1, 2, \dots, N\}$  which are not in  $\bigcup_{t=1}^l A_t(M)$ . Now notice that  $\Delta(N) \rightarrow \infty$ , hence  $A_M$  is infinite, which is a contradiction.

So,  $(n_k)_{k \in \mathbb{N}} \in P_1$  and the theorem is now proved.  $\square$

## References

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